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LETTER TO THE EDITOR

Vibrational resonance

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Abstract. The effect of a high-frequency force on the response of a bistable system to a low-frequency signal is considered for both the overdamped and weakly damped cases. It is shown that the response can be optimized by an appropriate choice of vibration amplitude. This *vibrational resonance* displays many analogies to the well known phenomenon of stochastic resonance, but with the vibrational force filling the role usually played by noise.

Stochastic resonance (SR) [1–3] is commonly said to occur when a weak periodic signal in a nonlinear system is amplified by an increase in the ambient noise intensity. In this Letter, we point out that an analogous phenomenon can occur when the noise is replaced by a high-frequency periodic force†. We will refer to it as *vibrational resonance* (VR). As we shall see, there are some interesting distinctions, as well as close analogies, between VR and SR.

SR is usually considered for the simplest possible example of an overdamped bistable oscillator described by the equation

$$\dot{x} + f(x) = A\cos\omega t + \xi(t) \tag{1}$$

where $A\cos\omega t$ is the weak input signal, $f(x) = \mathrm{d}U(x)/\mathrm{d}x$, $U(x)$ is a symmetric double-well potential (for example, $U(x) = -x^2/2 + x^4/4$), $\xi(t)$ is white noise of intensity K, i.e. $\langle\xi(t)\xi(t+\tau) = K\delta(\tau)$. The response of the system to the input signal can be considered either in terms of a linear susceptibility [6] or, as here, in terms of an effective stiffness [7]. SR occurs when the susceptibility or stiffness displays a nonmonotonic dependence on noise intensity, such that the response Q peaks at a particular value K_m of the noise intensity K. The complex amplitude B of the signal component $s_\omega(t)$ at frequency ω may be described by the linearized equation

$$\mathrm{i}\omega B + c(A, \omega, K)B = A \tag{2}$$

where $c(A, \omega, K)$ is a complex quantity. Its real part $c_\mathrm{r}(A, \omega, K)$ may be treated as an effective stiffness [7], whereas its imaginary part $c_\mathrm{i}(A, \omega, K)$ is proportional to what is effectively an additional damping factor. For SR to occur, it is essential that $c(A, \omega, K)$ should depend on the noise intensity K.

It is evident, however, that a change in $c(A, \omega, K)$ may be induced, not only by noise, but also by other kinds of high-frequency force such as, for example, a periodic vibration. Such a phenomenon would represent another example of a change in the dynamical behaviour

† We use the original definition of SR [4] in terms of signal amplification, rather the later definition [5] in terms of an enhancement of the signal-to-noise ratio.

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and properties of a nonlinear system induced by high-frequency vibration (see the book by Blekhman [8]). We first consider this possibility in relation to the overdamped bistable oscillator

$$\dot{x} - x + x^3 = A\cos\omega t + C\cos\Omega t \tag{3}$$

i.e. where the white noise $\xi(t)$ of (1) has been replaced by the rapidly oscillating periodic force $C\cos\Omega(t)$ with $\Omega \gg \omega$. We evaluate the response of the system to the input signal $A\cos\omega t$ (by analogy with a lock-in amplifier) by calculating the sine and cosine components, B_s and B_c respectively, of the coordinate variations (output signal), yielding

$$B_s = \frac{2}{nT}\int_0^{nT} x(t)\sin\omega t\,dt \qquad B_c = \frac{2}{nT}\int_0^{nT} x(t)\cos\omega t\,dt \tag{4}$$

where $T = 2\pi/\omega$ and n is an integer. Computing equation (3) and extracting the cosine and sine constituents of the output signal at frequency ω, we thus find the dependences on C of both the response amplitude $Q = \sqrt{B_s^2 + B_c^2}/A$ and the phase shift $\psi = -\arctan(B_s/B_c)$ of the response relative to the input signal. Some results for a fixed value of $\Omega = 5$ and different values of A and ω are plotted in figure 1.

It is immediately apparent from figures 1(*a*) and (*b*) that the form of $Q(C)$ is qualitatively similar to what is seen in SR [1–3]: the response Q at first increases with increasing amplitude C of the high-frequency force, but then passes through a maximum and decreases again. The maxima in $Q(C)$ are in general sharper than are found in the case of SR. The phase shifts $\Psi(C)$ are also reminiscent of those found in SR [9] but, again, exhibit very much sharper minima. For both $Q(C)$ and $\Psi(C)$, the positions of the extrema are weakly dependent on A and ω respectively, as shown in figure 1(*c*). The mechanism underlying VR can be understood as a reduction in the effective stiffness of the system induced by the high-frequency force [7], resulting in amplification of the low-frequency signal. Equivalently, it can be perceived in terms of changes in the depth of a smoothed auxiliary potential, as discussed earlier [10] for a quasimonochromatic force: here, signal amplification occurs when the low-frequency signal is first able to induce inter-well transitions; the nearly discontinuous rise in $Q(C)$ with increasing C corresponds to the minimum value of C at which the central maximum in the auxiliary potential can be destroyed at the peak signal values. The amplification decreases again when the amplitude of the high-frequency vibrational force has increased sufficiently to annihilate completely the double-well character of the auxiliary potential (and that of the real potential over most of each forcing period).

There is a clear analogy between SR and VR in the simple overdamped systems (1) and (3) respectively. We note that other authors have considered the effect of multi-frequency forces applied to nonlinear oscillators (see e.g. [11, 12]) but, to our knowledge, the phenomenon of vibrational resonance has not previously been identified or reported.

It is also of interest to establish whether VR arises in underdamped systems. As an example, we consider the weakly damped bistable oscillator

$$\ddot{x} + 2\delta\dot{x} - x + x^3 = A\cos\omega t + C\cos\Omega t. \tag{5}$$

One example of a system described by (5) is a pendulum placed between the opposite poles of a magnet [13]. The change of $c(A, \omega, C)$ with increasing amplitude of the high-frequency vibration means that the effective natural frequency of the oscillator described by equation (5) has to change too. It follows that the oscillator's response to the input signal $A\cos\omega t$ will depend on the amplitude of the high-frequency vibration—analogous to the 'tuning' of an underdamped monostable oscillator to resonance by adjustment of the noise intensity in the case of SR [14].

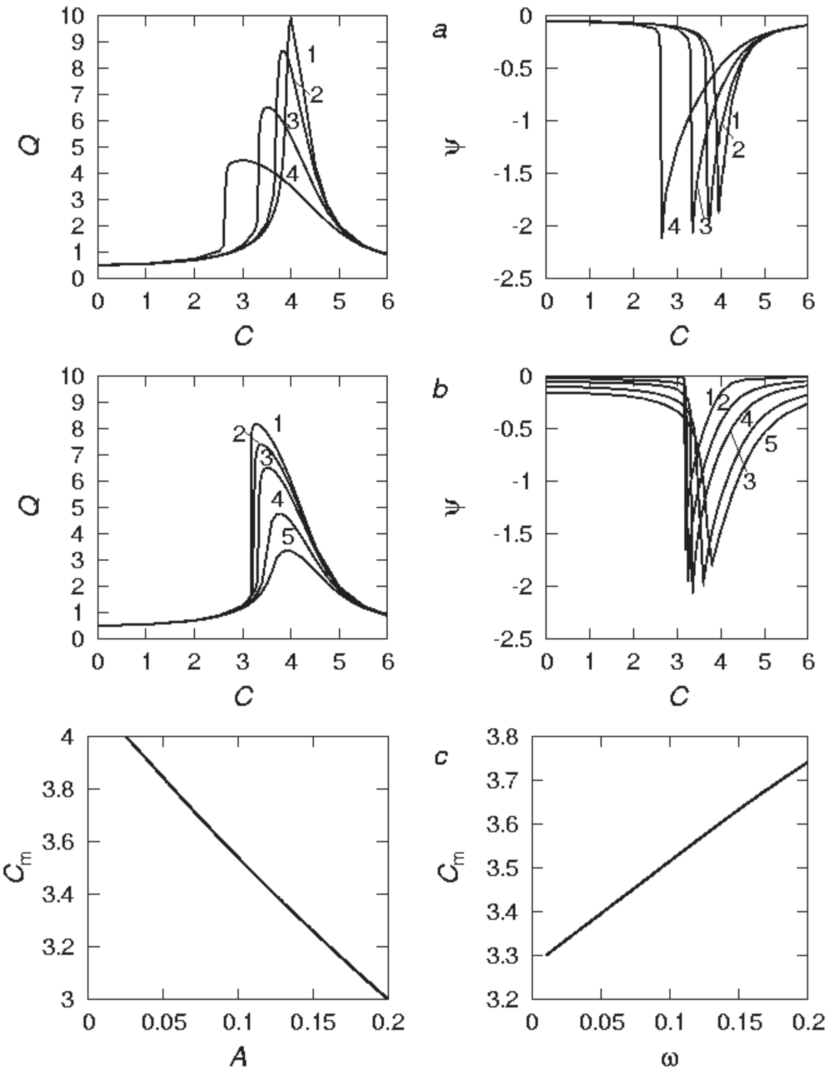

Figure 1. Response of the overdamped system (3) to the weak periodic signal $A\cos\omega t$, as influenced by the high-frequency vibrational force $C\cos\Omega t$ with $\Omega = 5$, for different conditions: (*a*) the response amplitude Q and corresponding phase shift ψ are plotted as functions of C for $\omega = 0.1$ with $A = 0.025$, $A = 0.05$, $A = 0.1$ and $A = 0.2$ (curves 1, 2, 3 and 4, respectively); (*b*) as in (*a*) but for $A = 0.1$ with $\omega = 0.01$, $\omega = 0.05$, $\omega = 0.1$, $\omega = 0.2$ and $\omega = 0.3$ (curves 1, 2, 3, 4 and 5, respectively); (*c*) plots of the characteristic vibration amplitudes $C_{\rm m}$ corresponding to maxima in $Q(C)$ as functions of the signal amplitude A (left) and ω (right)

Numerical simulation of equation (5) confirms these ideas. Some typical results are presented in figure 2 for different values of A and ω. The variations of the response amplitude $Q(C)$ and phase $\psi(C)$ with the amplitude C of the high-frequency vibrational force shown in figures 2(*a*) and (*b*) are markedly dependent on the signal frequency ω, unlike the case of the overdamped oscillator (3) (cf figures 1(*a*) and (*b*)).

For ω close to the frequency of small free oscillations there are two resonances (figure 2(*b*)), whereas for smaller ω there is only one resonance (figure 2(*a*)); the evolution with

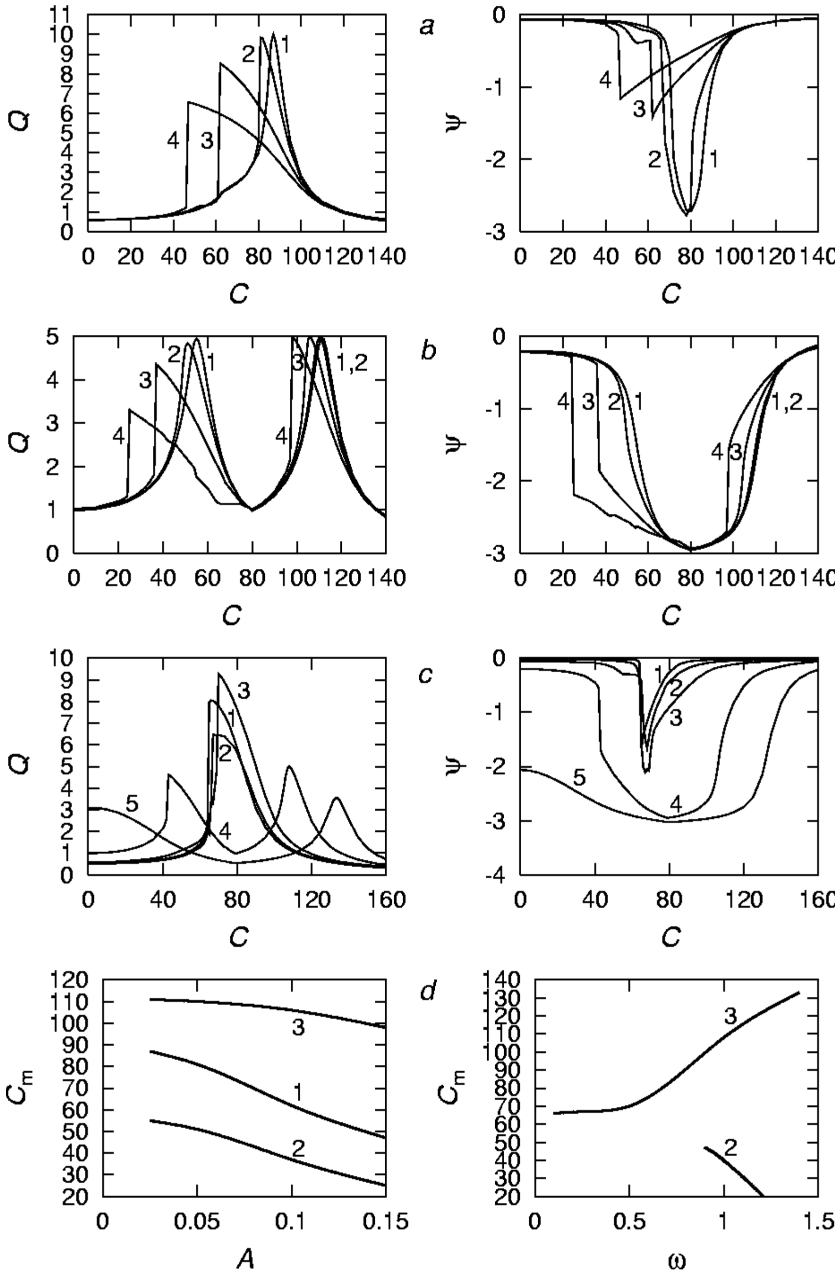

Figure 2. Response of the underdamped system (5) with $\delta = 0.1$ to the weak periodic signal $A\cos\omega t$, as influenced by the high-frequency vibrational force $C\cos\Omega t$ with $\Omega = 9.842$, for different conditions: (*a*) the response amplitude Q and corresponding phase shift ψ are plotted as functions of C with $\omega = 0.5$ for $A = 0.025$, $A = 0.05$, $A = 0.1$ and $A = 0.15$ (curves 1, 2, 3 and 4, respectively); (*b*) as in (*a*) but for $\omega = 1$; (*c*) as in (*a*) but for $A = 0.08$, $\omega = 0.1$, $\omega = 0.25$, $\omega = 0.5$, $\omega = 1$ and $\omega = 1.4$ (curves 1, 2, 3, 4 and 5, respectively); (*d*) plots of the characteristic vibration amplitudes C_m corresponding to maxima in $Q(C)$ as functions of signal amplitude A (left) for $\omega = 0.5$ (curve 1) and $\omega = 1$ (curves 2 and 3), and as functions of ω (right)

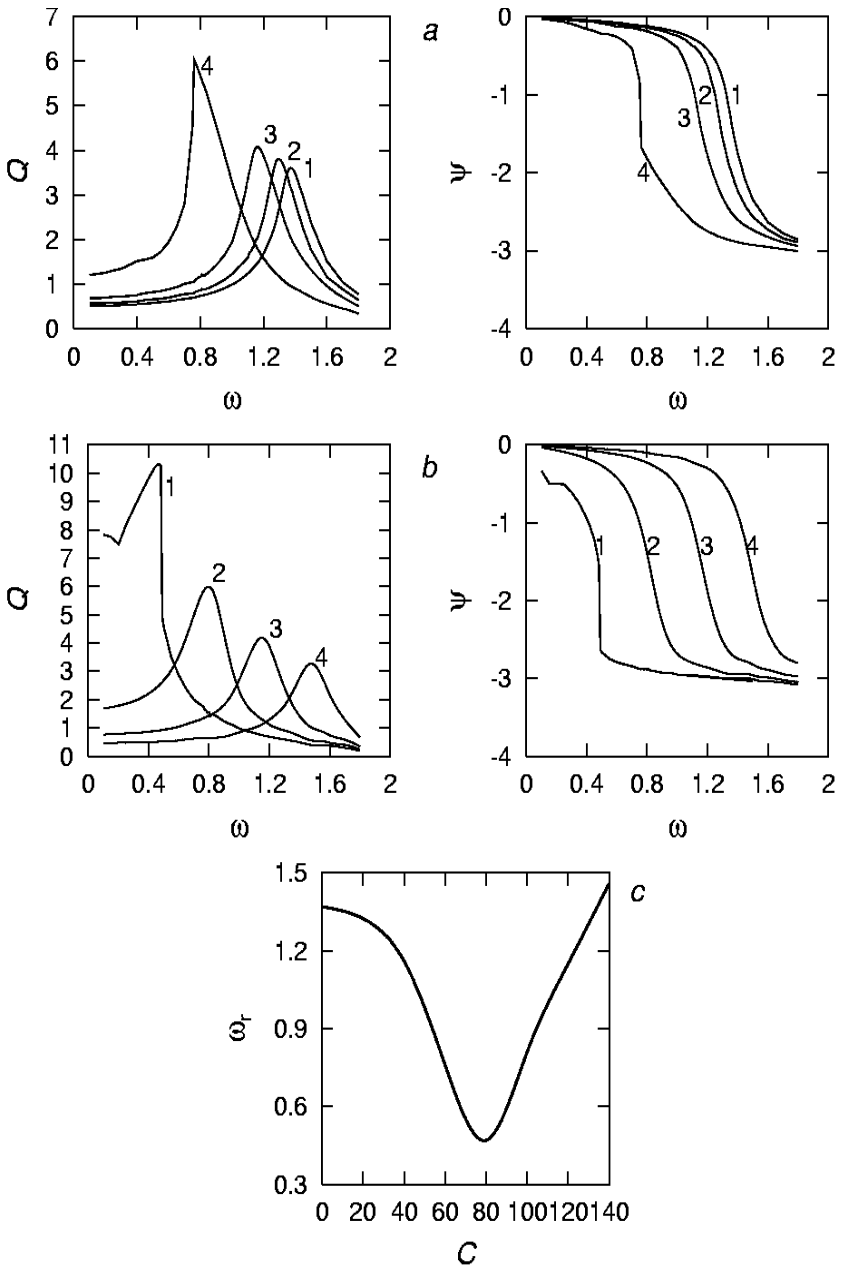

Figure 3. Dependences of the response amplitude Q and phase shift ψ on ω for $\delta = 0.1$, $\Omega = 9.842$, $A = 0.05$: (*a*) for vibration amplitudes $C = 0$, $C = 25$, $C = 40$ and $C = 60$ (curves 1, 2, 3 and 4, respectively); (*b*) for $C = 80$, $C = 100$, $C = 120$ and $C = 140$ (curves 1, 2, 3 and 4, respectively); (*c*) plot of the resonance frequency ω_r as a function of C

ω is seen more clearly in figure2(*c*) where a set of curves is plotted for different ω at fixed amplitude A. This phenomenon is attributable to the 'tuning' of two different oscillatory processes: oscillations inside an individual well, and those involving jumps between wells.

If the signal frequency is too far from the intra-well free resonance, the intra-well oscillation cannot be brought into resonance by the high-frequency vibration, and so only one maximum of $Q(C)$ is observed. Again, such behaviour is very similar to the case of stochastic resonance in a weakly damped bistable oscillator [7].

Figures 3(a) and (b) illustrate the resonant dependences of the response $Q(\omega)$ and phase shift $\psi(\omega)$ on ω for a fixed value of A and different values of C. In contrast to the overdamped oscillator (3), for a weakly damped oscillator these dependences are resonant in character, just as reported earlier for SR in the underdamped monostable oscillator [14]. As C increases, the resonant frequency at first decreases, but then increases again (see figure 3(c). Thus, we can control the resonant frequency $\omega_{\rm r}$ by changing the amplitude of the high-frequency vibration.

In conclusion, we have shown that the phenomenon of *vibrational resonance*, in which a weak periodic signal can be optimally amplified by the application of high-frequency periodic force of appropriate amplitude, can occur in both overdamped and underdamped nonlinear oscillators. It can be perceived as a form of stochastic resonance in which the noise has been replaced by a high-frequency periodic force.

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